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Transversals of Latin squares and covering radius of sets of permutations

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ABSTRACT

We consider the symmetric group $\mathcal{S}_n$ as a metric space with the Hamming metric. The covering radius $\mathrm{cr}(S)$ of a set of permutations $S \subset \mathcal{S}_n$ is the smallest r such that $\mathcal{S}_n$ is covered by the balls of radius r centred at the elements of S. For given n and s, let $f(n, s)$ denote the cardinality of the smallest set S of permutations with $\mathrm{cr}(S) \leqslant n - s$.

The value of $f(n, 2)$ is the subject of a conjecture by Kézdy and Snevily that implies two famous conjectures by Ryser and Brualdi on transversals in Latin squares. We show that $f(n, 2) \leqslant n + O(\log n)$ for all n and that $f(n, 2) \leqslant n + 2$ whenever $n = 3m$ for $m > 1$. We also construct, for each odd $m \geqslant 3$, a Latin square of order $3m$ with two rows that each contain $2m - 2$ transversal-free entries. This gives an infinite family of Latin squares with odd order n and at most $n/3 + O(1)$ disjoint transversals. The previous strongest upper bound for such a family was $n/2 + O(1)$.

Finally, we show that $f(5, 3) = 15$ and record a proof by Blackburn that $\mathrm{cr}(\mathrm{AGL}(1, q)) = q - 3$ when q is odd.

1. Introduction

Let S be a subset of a finite metric space, in which all the distances are integers. The *covering radius* $\mathrm{cr}(S)$ of S is the smallest r such that the balls of radius r with centres at the elements of S cover the whole space.

Consider the symmetric group $\mathcal{S}_n$ as a metric space equipped with *Hamming distance*. For any $g, h \in \mathcal{S}_n$, the *distance* between g and h is the number of points at which they disagree, i.e., n minus the number of fixed points of gh^{-1}. Note that it is invariant under left and right translation. The symmetric

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group has been studied as a setting for coding theory since the paper of Blake et al. [1] and in recent years some attention has been given to questions about covering radius [17].

1.1. The covering radius problem

The classical problem related to covering radius in coding theory is to find the smallest set with a given covering radius. In the case of the symmetric group, the problem is stated as: given n and s, what is the smallest m such that there is a set S of permutations with $|S| = m$ and $\mathrm{cr}(S) \leqslant n - s$? We let $f(n, s)$ denote this minimum value m.

By definition, $f(n, s)$ is a monotonic increasing function of s. Trivially, $f(n, 0) = 1$ and $f(n, n) = n!$. We also have $f(n, n-1) = n!$ for $n \geqslant 2$, since any two distinct permutations have distance at least 2. Recently, it was determined that $f(n, 1) = \lfloor n/2 \rfloor + 1$ in [3,4]. The next tempting case to consider is $f(n, 2)$. Kézdy and Snevily made the following conjecture.

Conjecture 1.1. *If n is even, then $f(n, 2) = n$; if n is odd, then $f(n, 2) > n$.*

The state of knowledge for small values of $f(n, 2)$ is provided in the following table. The value of $f(6, 2)$ is new, and was determined by exhaustive search. Others values are from [4].

n	3	4	5	6	7	8	9	10
$f(n, 2)$	6	4	6	6	$\leqslant 8$	$\leqslant 8$	$\leqslant 10$	$\leqslant 10$

The following general upper bounds are proved in [4].

Theorem 1.2.

$$f(n, 2) \leqslant \begin{cases} n & \text{if } n \text{ is even,} \\ \frac{5}{4}n + O(1) & \text{if } n \equiv 1 \bmod 4, \\ \frac{4}{3}n + O(1) & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

One of the primary aims of this paper is to present better upper bounds on $f(n, 2)$ when n is odd. Although we are unable to settle Conjecture 1.1, we will show for the first time that $f(n, 2)$ does not exceed a function that is asymptotically equal to n.

1.2. Transversals of Latin squares

A *Latin square of order* n is an $n \times n$ matrix L in which n distinct symbols are arranged so that each symbol occurs once in each row and column. We can specify L by a set of n^2 ordered triples $(x, y, z) \in \mathcal{I}(L)^3$, where $\mathcal{I}(L)$ is a set of cardinality n, and no two distinct triples agree in more than one coordinate. The interpretation is that z is the symbol in column y of row x. We say that L is *indexed* by $\mathcal{I}(L)$ and that (x, y, z) is an *entry* of L, i.e., $(x, y, z) \in L$. A *transversal* in a Latin square is a selection of n distinct entries in which each row, column and symbol is represented exactly once. See [18] for a recent survey on transversals of Latin squares, including their connections to covering radii of sets of permutations.

An entry that is not in any transversal will be described as *transversal-free*, whereas an entry that is in some transversal will be described as a *transversal entry*. In [10], computational evidence was given to suggest that Latin squares typically have a transversal through every entry.

Theorem 1.3 ([4]). *Let S be the set of n permutations corresponding to the rows of a Latin square L of order n. Then $\mathrm{cr}(S) = n - 1$ if L has a transversal and $\mathrm{cr}(S) = n - 2$ otherwise.*

It was conjectured by Ryser (see [6, p. 486]) that every Latin square of odd order has a transversal; this is still open. By Theorem 1.3, the Kézdy–Snevily Conjecture trivially implies Ryser's conjecture. In [6], Brualdi conjectured that every Latin square of order n contains a partial transversal of size $n - 1$

(see also [4,11,18]). It is known [4] that the Kézdy–Snevily Conjecture also implies Brualdi's conjecture. The fact that Conjecture 1.1 implies two famous unsolved conjectures provides significant motivation for its study.

For a given Latin square L of order n, we define $\lambda(L)$ to be the maximum m such that L has m disjoint transversals. It is well known that there are Latin squares of every even order with no transversals. In [9] and [10], it was shown that for even $n \geqslant 10$ and $j = 1$ or $j \equiv 0 \bmod 4$ such that $0 \leqslant j \leqslant n$, there exists a Latin square L of order n with $\lambda(L) = j$.

For all positive integers n, we define $\mu(n)$ to be the minimum value of λ among the Latin squares of order n. Clearly, $\mu(n) = 0$ for all even n, so we are concerned with the case when n is odd. If Ryser's Conjecture is true, then $\mu(n) \geqslant 1$ for all odd n. For $n \in \{1, 3\}$ we have $\mu(n) = n$. For order $n \in \{5, 7\}$ there is a Latin square whose transversals coincide on one entry, hence $\mu(5) = \mu(7) = 1$. By computation, it was determined in [9] that $\mu(9) = 3$. A general upper bound for $\mu(n)$, derived from [12,14] and first stated in [10], is the following.

Theorem 1.4. *If n is odd and $n > 3$, then $\mu(n) \leqslant \frac{1}{2}(n + 1)$.*

1.3. Our contributions

In this paper, we focus primarily on the investigation of the values $f(n, 2)$. We improve the upper bounds in Theorem 1.2 by giving explicit sets of permutations in $\mathcal{S}_n$ with covering radius $n - 2$. In particular, we show that $f(n, 2) \leqslant n + O(\log n)$ for all n. For $n > 3$ and divisible by 3, we prove a better bound saying that $f(n, 2) \leqslant n + 2$. Further, we improve the bound in Theorem 1.4 for odd n divisible by 3, by showing that $\mu(n) \leqslant \frac{1}{3}n + 2$ in this case.

Additionally, we determine the exact value of $f(5, 3)$, which equals 15. This is the smallest non-trivial case for $f(n, 3)$. We also give a proof due to Blackburn that the covering radius of $\mathrm{AGL}(1, q)$ is $q - 3$ when q is odd.

2. A general upper bound

In this section, we define a family of Latin squares $\mathcal{H}_n$ with the property that each $\mathcal{H}_n$ has two transversal-free entries in the second row and strong restrictions on the transversal entries in the first and third rows. This family of Latin squares was first constructed by Egan [7] to prove the existence of Latin squares with large indivisible plexes. Here we use $\mathcal{H}_n$ to prove that $f(n, 2) \leqslant n + O(\log n)$ for all odd n. However, we first introduce some useful notation.

Let $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be a set of size n and consider the natural action of $\mathcal{S}_n$ on Σ. For each $g \in \mathcal{S}_n$ and $1 \leqslant i \leqslant n$, we write σ_i^g for the image of σ_i under g. The passive form of g is the word $\sigma_1^g \sigma_2^g \cdots \sigma_n^g$. Let L be a Latin square indexed by Σ. We will associate each transversal $T = \{(x_i, \sigma_i, z_i) : i = 1, 2, \ldots, n\}$ of L with the corresponding permutation $z_1 z_2 \cdots z_n$ in $\mathcal{S}_n$.

Next, we sketch the main method of constructing a set $S \subset \mathcal{S}_n$ with covering radius $n - 2$ used throughout the paper. Let S' be the set of n permutations corresponding to the rows of the Latin square L. Note that the permutations not covered by the balls of radius $n - 2$ with centres at the elements of S' are those corresponding to the transversals of L. Find a small set $S'' \subset \mathcal{S}_n$ whose balls of radius $n - 2$ cover all the permutations corresponding to the transversals, then $S = S' \cup S''$ is the desired set with covering radius $n - 2$. Since the cardinality of S'' determines that of S, we want S'' as small as possible, which is achieved by constructing the Latin square L with strongly restricted transversals.

Now we state a key lemma, which is extremely useful for studying transversals in a Latin square. Let G be an Abelian group and let L be a Latin square of order $|G|$ where $\mathcal{I}(L) = G$. For each entry $e = (x, y, z)$ of L, define the function $\Delta : L \to G$ by $\Delta(e) = z - x - y$.

Lemma 2.1. *Let G be an Abelian group with identity ε and let L be a Latin square indexed by G. If T is a transversal in L, then*

$$\sum_{e \in T} \Delta(e) = -\sum_{g \in G} g = \begin{cases} \omega & \text{if } G \text{ has a unique involution } \omega, \\ \varepsilon & \text{otherwise.} \end{cases}$$

Variations of Lemma 2.1 can be found in [2,5,7–10,12,16,19]. When applying this lemma, we focus on the following subsets of L:

$$\Delta_* = \{e \in L : \Delta(e) \neq \varepsilon\}, \qquad \Delta_g = \{e \in L : \Delta(e) = g\},$$

where g is an element of $G \setminus \{\varepsilon\}$.

2.1. Latin squares $\mathcal{H}_n$

Latin square $\mathcal{H}_n$: For odd $n > 3$, we define the Latin square $\mathcal{H}_n$ of order n and indexed by $\mathbb{Z}_n$ as follows, where $\mathcal{H}_n[x, y] = z$ for each $(x, y, z) \in \mathcal{H}_n$. Let $F = \{1, 3, 5, \ldots, n-2\} \subset \mathbb{Z}_n$ and $E = \mathbb{Z}_n \setminus F$.

$$\mathcal{H}_n[x, y] = \begin{cases} 1 & \text{if } (x, y) \in \{(0, 0), (1, n-1)\}, \\ 0 & \text{if } (x, y) \in \{(1, 0), (2, n-1)\}, \\ y+2 & \text{if } x = 0 \text{ and } y \in F, \\ y & \text{if } x = 2 \text{ and } y \in F, \\ x+y & \text{otherwise.} \end{cases}$$

For each $e \in \Delta_* \subset \mathcal{H}_n$, the value of $\Delta(e)$ is given below.

	0	1	3	$\cdots$	$n-2$	$n-1$
0	1	2	2	$\cdots$	2	
1	-1					1
2		-2	-2	$\cdots$	-2	-1

It has been proved [7] that the entries $(1, 0, 0)$ and $(1, n-1, 1)$ in $\mathcal{H}_n$ are not in any transversal. Now we consider transversal entries in $\mathcal{H}_n$. If T is a transversal of $\mathcal{H}_n$ including an entry in Δ_2, then by Lemma 2.1, T must include an entry in Δ_{-2}, i.e., the two transversal entries of T in rows 0 and 2 are in two distinct columns from F. Similarly, if T includes an entry in row 0 and column from $E \setminus \{0\}$, then the corresponding transversal entry in row 2 must be in a distinct column from $E \setminus \{n-1\}$. Further, $(0, 0, 1) \in T$ if and only if $(2, n-1, 0) \in T$. These simple observations will allow us to find a better upper bound on $f(n, 2)$.

2.2. Biclique cover

Let $G = (V, E)$ be a simple undirected graph. A *biclique cover* of G is a collection of bicliques (complete bipartite subgraphs) of G such that every edge of G belongs to at least one of these bicliques. The *biclique covering number* $bc(G)$ of G is the cardinality of a minimum biclique cover (MBC) of G; we use the notation of [15] (this parameter is called the *bipartite dimension* in [13]). The MBC problem is the problem of determining $bc(G)$ for any simple graph G.

Now we consider the MBC problem for a family of bipartite graphs. Let $K'_{m,m}$ be a bipartite graph with two vertex parts $A_i = \{1_i, 2_i, \ldots, m_i\}$, $i = 1, 2$. For all $k, l \in \{1, 2, \ldots, m\}$, k_1 is adjacent to l_2 except when $l = k$ or $l = k+1$. Note that the calculations in this subsection are all in $\mathbb{Z}$. Denote $bc(K'_{m,m})$ by $bc(m)$ for short. Trivially, $bc(1) = 0$ and $bc(2) = 1$. For convenience, we describe a biclique B in $K'_{m,m}$ by specifying the vertices that induce B, writing $B = B_1|B_2$, where $B_i \subset A_i$, $i = 1, 2$.

Example 2.2. Here is a biclique cover of $K'_{4,4}$ of cardinality four comprised of $P, Q, B^{(1)}$ and $B^{(2)}$, where

$$P = \{1_1, 2_1\}|\{4_2\}, \qquad Q = \{3_1, 4_1\}|\{1_2, 2_2\},$$
$$B^{(1)} = \{1_1, 4_1\}|\{3_2\} \quad \text{and} \quad B^{(2)} = \{2_1\}|\{1_2\}.$$

When $m = 2^k$, we have the following upper bound on the biclique covering number of $K'_{m,m}$.

Lemma 2.3. *$bc(2^k) \leqslant 3k - 2$ for all positive integers k.*

Proof. It is true when $k = 1$. For each subset $X \subseteq \{1_i, 2_i, 3_i, \ldots, 2^k_i\}$, $i = 1, 2$, denote $\hat{X} = \{(a + 2^k)_i : a_i \in X\}$. We prove the upper bound on $bc(2^k)$ by giving a constructive algorithm for a biclique cover of $K'_{2^k,2^k}$ with cardinality $3k-2$ for all $k \geqslant 2$, which involves the following steps.

(S.1) Initialize $k=2$. Let P, Q and a set $\mathcal{B} = \{B^{(1)}, B^{(2)}\}$ be the biclique cover of $K'_{4,4}$ of size 4 listed in Example 2.2.

(S.2) Suppose that $\{P, Q\} \cup \mathcal{B}$ is a biclique cover of $K'_{2^k,2^k}$ of size $3k-2$ with $|\mathcal{B}| = 3k-4$. We will construct a biclique cover of $K'_{2^{k+1},2^{k+1}}$ of size $3k+1$, including P', Q' and a set $\mathcal{B}'$ with $|\mathcal{B}'| = 3k-1$.

  (i) Define two bicliques $P' = \{1_1, 2_1, \ldots, 2^k_1\}|\{(2^k+2)_2, (2^k+3)_2, \ldots, 2^{k+1}_2\}$ and $Q' = \{(2^k+1)_1, (2^k+2)_1, \ldots, 2^{k+1}_1\}|\{1_2, 2_2, \ldots, 2^k_2\}$.

  (ii) For each $B \in \mathcal{B}$, define a biclique $B' = (B_1 \cup \hat{B_1})|(B_2 \cup \hat{B_2})$ of $K'_{2^{k+1},2^{k+1}}$. Put B' into $\mathcal{B}'$.

  (iii) Let $B'_P = (P_1 \cup \hat{Q_1})|(P_2 \cup \hat{Q_2})$ and $B'_Q = (\hat{P_1} \cup Q_1)|(\hat{P_2} \cup Q_2)$. Put B'_P and B'_Q into $\mathcal{B}'$.

  (iv) Finally, put a biclique $B'_F = \{1_1, 2_1, \ldots, (2^k-1)_1\}|\{(2^k+1)_2\}$ into $\mathcal{B}'$ and go to (S.3).

(S.3) It is easy to check that $\{P', Q'\} \cup \mathcal{B}'$ is a biclique cover of $K'_{2^{k+1},2^{k+1}}$ of size $3k+1$. Update $k := k+1$, $P := P'$, $Q := Q'$ and $\mathcal{B} := \mathcal{B}'$, then go to (S.2).

From the above algorithm, we obtain a biclique cover of $K'_{2^k,2^k}$ of size $3k-2$ for all $k \geqslant 2$. □

Example 2.4. Applying step (S.2) in Lemma 2.3 to the biclique cover of $K'_{4,4}$ in Example 2.2, we get a biclique cover of $K'_{8,8}$ of size seven consisting of the bicliques below.

$$P' = \{1_1, 2_1, 3_1, 4_1\}|\{6_2, 7_2, 8_2\}, \qquad Q' = \{5_1, 6_1, 7_1, 8_1\}|\{1_2, 2_2, 3_2, 4_2\},$$
$$B'^{(1)} = \{1_1, 4_1, 5_1, 8_1\}|\{3_2, 7_2\}, \qquad B'^{(2)} = \{2_1, 6_1\}|\{1_2, 5_2\},$$
$$B'_P = \{1_1, 2_1, 7_1, 8_1\}|\{4_2, 5_2, 6_2\}, \qquad B'_Q = \{3_1, 4_1, 5_1, 6_1\}|\{1_2, 2_2, 8_2\} \quad \text{and}$$
$$B'_F = \{1_1, 2_1, 3_1\}|\{5_2\}.$$

Observing that $K'_{m,m}$ is an induced subgraph of $K'_{l,l}$ if $m \leqslant l$, we get the following corollary of Lemma 2.3.

Corollary 2.5. *$bc(m) \leqslant 3\lceil \log_2 m \rceil - 2$ for all integers $m > 1$.*

Proof. Let k be the smallest integer such that $m \leqslant 2^k$. By Lemma 2.3, $K'_{2^k,2^k}$ has a biclique cover of size $3k-2$, which induces a cover of $K'_{m,m}$. □

2.3. An upper bound on $f(n, 2)$

We begin by listing the entries of $\mathcal{H}_n$ in rows 0 and 2 in the following table.

	0	1	2	3	4	5	6	···	$n-6$	$n-5$	$n-4$	$n-3$	$n-2$	$n-1$
0	1	(3	2)	(5	4)	(7	6)	···	($n-4$	$n-5$)	($n-2$	$n-3$)	(0	$n-1$)
2	(2	1)	(4	3)	(6	5)	(8	···	$n-6$)	($n-3$	$n-4$)	($n-1$	$n-2$)	0

(1)

Regard each pair of entries $\{(r, c, s_1), (r, c+1, s_2)\}$ within parentheses in (1) as a vertex. Let A_1 be the vertex set defined in row 0 ordered in a natural way, and A_2 be that in row 2. Obviously, $|A_1| = |A_2| = (n-1)/2$. Let Γ be the bipartite graph $K'_{(n-1)/2,(n-1)/2}$ defined over $A_1 \cup A_2$ as in Section 2.2. Note that if two vertices $u = \{(0, c, s_1), (0, c+1, s_2)\} \in A_1$ and $v = \{(2, c', s'_1), (2, c'+1, s'_2)\} \in A_2$ are not adjacent in Γ, then no transversal of $\mathcal{H}_n$ includes one entry from u and one from v simultaneously. By Corollary 2.5, Γ has a biclique cover of size $3\lceil \log_2(n-1) \rceil - 5$, which consists of P, Q, and a set $\mathcal{B}$.

Lemma 2.6. *$f(n, 2) \leqslant n + 3\lceil \log_2(n-1) \rceil - 5$ for all odd $n > 3$.*

Proof. We have to construct a set S of $n + 3\lceil \log_2(n-1) \rceil - 5$ permutations which have at least two agreements with every permutation in $\mathcal{S}_n$. The first n permutations are from the rows of $\mathcal{H}_n$. The

	00	01	02	$\cdots$	0ℓ	10	11	12	$\cdots$	1ℓ	20	21	22	$\cdots$	2ℓ
00	10	10	10	$\cdots$	10	20	20	20	$\cdots$	20					
01	0ℓ	0ℓ	0ℓ	$\cdots$	0ℓ										
02															
$\vdots$															
0ℓ															
10	21	21	21	$\cdots$	21						11	11	11	$\cdots$	11
11															
12															
$\vdots$															
1ℓ															
20						10	10	10	$\cdots$	10	20	20	20	$\cdots$	20
21											01	01	01	$\cdots$	01
22											01	01	01	$\cdots$	01
$\vdots$											$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
2ℓ											01	01	01	$\cdots$	01

Fig. 1. Values of Δ on $\mathcal{D}_{3m}$.

remaining permutations of S are based on the biclique cover $\{P, Q\} \cup \mathcal{B}$ of Γ from Corollary 2.5. For each biclique $B \in \mathcal{B} \cup \{Q\}$, define g_B to be any permutation over $\mathbb{Z}_n$ satisfying that $c^{g_B} = s$ for all entries (r, c, s) involved in the vertices of B. The permutation g_B is well defined since for any two entries (r, c, s) and (r', c', s') contained in the vertices of B, we have $c \neq c'$ and $s \neq s'$. Finally, define g_P to be a permutation such that $0^{g_P} = 1$, $(n-1)^{g_P} = 0$ and $c^{g_P} = s$ for all entries (r, c, s) involved in the vertices of P. By construction, each permutation corresponding to a transversal of $\mathcal{H}_n$ agrees in at least two places with a permutation g_B for at least one $B \in \{P, Q\} \cup \mathcal{B}$. □

3. A better upper bound for orders divisible by three

In this section we use the following family of Latin squares $\mathcal{D}_{3m}$, which were first constructed in [10], to give the upper bound $f(3m, 2) \leqslant 3m + 2$ for all odd $m \geqslant 3$. Note that in both this and the next section, we consider $\mathcal{S}_{3m}$ acting on $\mathbb{Z}_3 \oplus \mathbb{Z}_m$, which we will always write in the order

$$(0, 0), (0, 1), \ldots, (0, m-1), (1, 0), (1, 1), \ldots, (1, m-1), (2, 0), (2, 1), \ldots, (2, m-1).$$

Latin square $\mathcal{D}_{3m}$: For odd $m \geqslant 3$, we define the Latin square $\mathcal{D}_{3m}$ of order $3m$ and indexed by $\mathbb{Z}_3 \oplus \mathbb{Z}_m$.

$$\mathcal{D}_{3m}[(a, b), (c, d)] = \begin{cases} (1, d) & \text{if } (a = b = c = 0) \text{ or } (a = 2, b = 0 \text{ and } c = 1), \\ (0, d) & \text{if } (a = b = 0 \text{ and } c = 1) \text{ or } (a = c = 2 \text{ and } b = 0), \\ (0, d+1) & \text{if } a = 1 \text{ and } b = c = 0, \\ (1, d+1) & \text{if } a = 1, b = 0 \text{ and } c = 2, \\ (0, d) & \text{if } a = c = 0 \text{ and } b = 1, \\ (1, b+d+1) & \text{if } a = c = 2 \text{ and } b \neq 0, \\ (a+c, b+d) & \text{otherwise.} \end{cases}$$

Let $\ell = m - 1$. For each $e \in \Delta_* \subset \mathcal{D}_{3m}$, the abbreviated ordered pairs $\Delta(e)$ are displayed in Fig. 1.

We use the same notation as in [10]. Suppose that T is a transversal of $\mathcal{D}_{3m}$. Define x_{ij} to be the number of entries in T of the form $\big((i, b), (j, d), (e, f)\big)$ where b, d and f are arbitrary and $e = i + j$ in $\mathbb{Z}_3$, and y_{ij} to be the number of entries in T of the same form, but where $e \neq i + j$ in $\mathbb{Z}_3$. Finally, let z be the number of entries in T of the form $\big((0, 1), (0, d), (0, d)\big)$ where d is arbitrary. The shaded region in Fig. 1 shows an $(m-1) \times m$ subrectangle, which has been proved to consist of transversal-free entries [10].

Lemma 3.1 (*[10]*). $y_{10} = x_{00} - z = 0$.

A number of constraints are immediate from the definition of $\mathcal{D}_{3m}$, such as $0 \leqslant x_{ij} \leqslant m, 0 \leqslant y_{ij} \leqslant 1$, $0 \leqslant z \leqslant 1$ and the fact that $y_{02} = y_{11} = y_{20} = 0$. Moreover, the construction of $\mathcal{D}_{3m}$ forces

$$y_{00} + y_{01} \leqslant 1, \qquad y_{21} + y_{22} \leqslant 1, \tag{2}$$
$$0 \leqslant x_{22} \leqslant m - 1. \tag{3}$$

Also, the need for T to include one representative from each row, column and symbol of $\mathcal{D}_{3m}$ implies

$$x_{00} + x_{01} + x_{02} + y_{00} + y_{01} = m, \tag{4}$$
$$x_{20} + x_{21} + x_{22} + y_{21} + y_{22} = m, \tag{5}$$
$$x_{01} + x_{11} + x_{21} + y_{01} + y_{21} = m, \tag{6}$$
$$x_{02} + x_{12} + x_{22} + y_{12} + y_{22} = m, \tag{7}$$
$$x_{00} + x_{12} + x_{21} + y_{01} + y_{22} = m, \tag{8}$$
$$x_{02} + x_{11} + x_{20} = m. \tag{9}$$

Adding (4), (5), then subtracting (6) and (9), gives

$$x_{00} + x_{22} - 2x_{11} + y_{00} + y_{22} = 0. \tag{10}$$

Adding (5), (7), then subtracting (8) and (9), gives

$$2x_{22} - x_{00} - x_{11} + y_{12} + y_{21} + y_{22} - y_{01} = 0. \tag{11}$$

Moreover, Lemma 2.1 necessitates that

$$3 \mid y_{00} + 2y_{01} + y_{12} + y_{21} + 2y_{22}, \tag{12}$$
$$m \mid x_{22} + y_{12} - z. \tag{13}$$

Define u to be the number of entries in T of the form $\big((1, 0), (1, d), (2, d)\big)$ where d is arbitrary. Then $u + y_{12} = 1$ since $y_{10} = 0$.

Lemma 3.2. *If $z = 0$, then $u = y_{00} = y_{22} = 1$.*

Proof. We assume that $z = 0$. Lemma 3.1 implies that $x_{00} = 0$.

First suppose that $u = 0$. Then $y_{12} = 1$, which with (13) leads to $x_{22} = m - 1$. By (11),

$$\begin{aligned} x_{11} &= 2x_{22} - x_{00} + y_{12} + y_{21} + y_{22} - y_{01} \\ &= 2(m-1) + 1 + y_{21} + y_{22} - y_{01} \\ &\geqslant 2(m-1) + 1 + 0 - 1 = 2(m-1). \end{aligned}$$

However, $u = 0$ implies $x_{11} \leqslant m - 1$ which yields a contradiction.

So $u = 1$, which implies $y_{12} = 0$. By (13), it follows that $x_{22} = 0$, which together with (10) gives $y_{00} + y_{22} = 2x_{11} \geqslant 2u = 2$. Hence, $y_{00} = y_{22} = 1$. □

Note that Lemma 3.2 implies $z + y_{00} \geqslant 1$, which means that each transversal of $\mathcal{D}_{3m}$ includes at least one entry in rows $(0, 0)$ or $(0, 1)$, and column $(0, d)$, for some d. Further, if a transversal includes no entry in row $(0, 1)$ and column $(0, d)$, then it must include an entry in row $(1, 0)$ and column $(1, e)$ for some e. This allows us to give an upper bound for $f(3m, 2)$.

Theorem 3.3. *$f(3m, 2) \leqslant 3m + 2$ for all odd $m \geqslant 3$.*

Proof. We take as our set S the rows of $\mathcal{D}_{3m}$, plus 2 further permutations

$$(0,0)(0,1)\cdots(0,l)(2,0)(2,1)\cdots(2,l)(1,1)(1,2)\cdots(1,l)(1,0)$$

and

$$(1,0)(1,1)\cdots(1,l)(2,0)(2,1)\cdots(2,l)(0,0)(0,1)\cdots(0,l).$$

The first ensures two agreements with the permutations corresponding to any transversal with $z = 1$, given that $u + y_{12} = 1$. The second ensures three agreements with those permutations corresponding to any transversal with $z = 0$, given that $y_{00} = u = y_{22} = 1$. □

4. Latin squares with many transversal-free entries in a row

In this section, we construct a family of Latin squares $\mathcal{F}_{3m}$, where for each odd $m \geqslant 3$, $\mathcal{F}_{3m}$ has order $3m$ with two rows that each contain $2m-2$ transversal-free entries.

4.1. Latin squares $\mathcal{F}_{3m}$

For distinct $i, j \in \{0, 1, 2, 3\}$, define $\Psi_{i,j} \subseteq \mathbb{Z}_m$ to be the set of congruence classes modulo m that contain an integer in the interval $[0, m-1]$ that is congruent to i or j modulo 4. Given an element $c \in \mathbb{Z}_m$ and a subset $A \subseteq \mathbb{Z}_m$, define $A + c = \{a + c \in \mathbb{Z}_m : a \in A\}$. For each $i \in \mathbb{Z}_3$, we define subsets Y_i, U_i, V_i and W_i of $\mathbb{Z}_m$ as follows:

$$\begin{array}{ll} Y_0 = Y_2 = \Psi_{2,3}, & Y_1 = Y_0 + 1, \\ U_0 = U_2 = \Psi_{0,1}, & U_1 = \mathbb{Z}_m \setminus Y_1, \\ V_0 = V_2 = Y_0 + 2, & V_1 = Y_0 - 1, \\ W_0 = W_2 = U_0 \setminus V_0, \quad \text{and} & W_1 = U_1 \setminus V_1. \end{array}$$

Lemma 4.1. *The subsets Y_i, U_i, V_i and W_i of $\mathbb{Z}_m$ defined above satisfy the following properties:*
(i) $V_i = \mathbb{Z}_m \setminus (U_1 + 1)$ *for* $i = 0, 2$;
(ii) $|V_i| = |Y_i| = (m-1)/2$, $|U_i| = (m+1)/2$ *and* $|W_i| = 1$, *for each* $i \in \mathbb{Z}_3$;
(ii) V_i *and* W_i *partition* U_i, *whereas* U_i *and* Y_i *partition* $\mathbb{Z}_m$, *for each* $i \in \mathbb{Z}_3$.

Proof. We give the proof by explicitly listing the subsets Y_i, U_i, V_i and W_i, $i \in \mathbb{Z}_3$, which we split into two cases. When $m \equiv 1 \bmod 4$, the subsets are

$$\begin{array}{ll} Y_0 = Y_2 = \Psi_{2,3}, & Y_1 = \Psi_{0,3} \setminus \{0\}, \\ U_0 = U_2 = \Psi_{0,1}, & U_1 = \Psi_{1,2} \cup \{0\}, \\ V_0 = V_2 = \Psi_{0,1} \setminus \{1\}, & V_1 = \Psi_{1,2}, \\ W_0 = W_2 = \{1\}, \quad \text{and} & W_1 = \{0\}. \end{array}$$

When $m \equiv 3 \bmod 4$, the subsets are

$$\begin{array}{ll} Y_0 = Y_2 = \Psi_{2,3}, & Y_1 = \Psi_{0,3}, \\ U_0 = U_2 = \Psi_{0,1}, & U_1 = \Psi_{1,2}, \\ V_0 = V_2 = \Psi_{0,1} \setminus \{0\}, & V_1 = \Psi_{1,2} \setminus \{-1\}, \\ W_0 = W_2 = \{0\}, \quad \text{and} & W_1 = \{-1\}. \end{array}$$

Properties (i)–(iii) are routine to check in each case. □

Now we are in a position to construct our Latin squares.

Latin square $\mathcal{F}_{3m}$: For odd $m \geqslant 3$, we define the Latin square $\mathcal{F}_{3m}$ of order $3m$ and indexed by $\mathbb{Z}_3 \oplus \mathbb{Z}_m$. Let $\ell = m - 1$ and $k = m - 2$ and define

$$\mathcal{F}_{3m}[(a, b), (c, d)] = \begin{cases} (a + c, b + d - 1) & \text{if } (a = c = 0 \text{ or } a = c = 2) \text{ and } b \neq \ell, \\ (c + 1, d) & \text{if } (a, b) = (0, \ell) \text{ and } (c = 0, d \in U_0 \text{ or } c = 2, d \in Y_2), \\ (c + 2, d - 1) & \text{if } (a, b) = (0, \ell) \text{ and } (c = 0, d \in Y_0 \text{ or } c = 1, d \in U_1), \\ (c + 1, d) & \text{if } (a, b) = (2, \ell) \text{ and } (c = 0, d \in Y_0 \text{ or } c = 2, d \in U_2), \\ (c, d - 1) & \text{if } (a, b) = (2, \ell) \text{ and } (c = 1, d \in U_1 \text{ or } c = 2, d \in Y_2), \\ (0, d - 2) & \text{if } (a, b) = (1, 0) \text{ and } c = 0, \\ (1, d - 2) & \text{if } (a, b) = (1, 0) \text{ and } c = 2, \\ (a + c, b + d) & \text{otherwise.} \end{cases}$$

For each $e \in \Delta_* \subset \mathcal{F}_{3m}$, we display in Fig. 2 the abbreviated ordered pairs $\Delta(e)$. For convenience, we rearrange the columns corresponding to the subsets W_i, V_i and Y_i of $\mathbb{Z}_m$, $i \in \mathbb{Z}_3$. The shaded regions will be shown to consist of transversal-free entries.

First, we will show that $\mathcal{F}_{3m}$ has a twofold symmetry. An *isotopy* of a Latin square L is a permutation of its rows, a permutation of its columns, and a permutation of its symbols. An isotopy that maps L to itself is called an *autotopy* of L.

	$\{0\}\oplus$							$\{1\}\oplus$							$\{2\}\oplus$						
	W_0	V_0			Y_0			W_1	V_1			Y_1			W_2	V_2			Y_2		
00	0ℓ	0ℓ	$\cdots$	0ℓ	0ℓ	$\cdots$	0ℓ														
01	0ℓ	0ℓ	$\cdots$	0ℓ	0ℓ	$\cdots$	0ℓ														
$\vdots$		$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$														
$0k$	0ℓ	0ℓ	$\cdots$	0ℓ	0ℓ	$\cdots$	0ℓ														
0ℓ	11	11	$\cdots$	11	20	$\cdots$	20	20	20	$\cdots$	20								11	$\cdots$	11
10	$2k$	$2k$	$\cdots$	$2k$	$2k$	$\cdots$	$2k$								$1k$	$1k$	$\cdots$	$1k$	$1k$	$\cdots$	$1k$
11																					
$\vdots$																					
1ℓ																					
20															0ℓ	0ℓ	$\cdots$	0ℓ	0ℓ	$\cdots$	0ℓ
21															0ℓ	0ℓ	$\cdots$	0ℓ	0ℓ	$\cdots$	0ℓ
$\vdots$															$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$2k$															0ℓ	0ℓ	$\cdots$	0ℓ	0ℓ	$\cdots$	0ℓ
2ℓ					21	$\cdots$	21	10	10	$\cdots$	10				21	21	$\cdots$	21	10	$\cdots$	10

Fig. 2. Values of Δ on $\mathcal{F}_{3m}$.

Let $\sigma = (0\ 1)$ and $\tau = (0\ 2)$ be two transpositions acting on $\mathbb{Z}_3$. Define a permutation α of $Z_3 \oplus Z_m$ by $(a, b)^\alpha = (a^\tau, b)$ and a permutation β of $Z_3 \oplus Z_m$ by $(a, b)^\beta = (a^\sigma, b)$. Let $\mathcal{F}'_{3m}$ be the Latin square obtained by applying α to the rows and columns of $\mathcal{F}_{3m}$, and β to the symbols.

Lemma 4.2. $\mathcal{F}'_{3m} = \mathcal{F}_{3m}$.

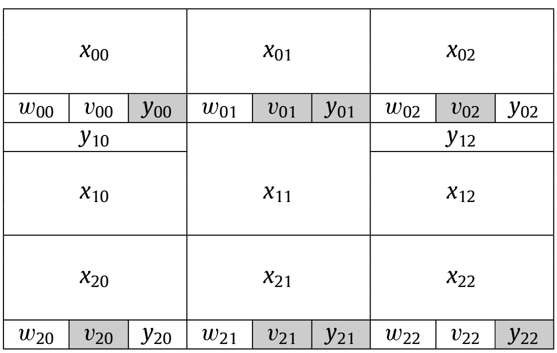
Proof. It is routine to verify that $\mathcal{F}'_{3m}[(a, b), (c, d)] = \mathcal{F}_{3m}[(a, b), (c, d)]$ holds for all $(a, b), (c, d)$ in $\mathbb{Z}_3 \oplus \mathbb{Z}_m$. For example, when $a = 0$, $b = -1$ and $c = 0$, $d \in Y_0$,

$$\begin{aligned}\mathcal{F}'_{3m}[(a, b), (c, d)] &= \mathcal{F}'_{3m}[(0, -1), (0, d)] \\ &= \mathcal{F}_{3m}[(2, -1), (2, d)]^\beta = (2, d-1)^\beta \\ &= (2, d-1) = \mathcal{F}_{3m}[(a, b), (c, d)].\end{aligned}$$

The third equality holds since $Y_0 = Y_2$. □

4.2. Transversal-free entries in $\mathcal{F}_{3m}$

We partition $\mathcal{F}_{3m}$ into subrectangles as follows according to the distinct $\Delta(e)$ values shown in Fig. 2 and the subsets W_i, V_i and Y_i of $\mathbb{Z}_m$, $i \in \mathbb{Z}_3$. The short boxes represent segments in rows 0ℓ, 10 and 2ℓ respectively. We also shade the regions that will be shown to consist of transversal-free entries.



Suppose that T is a transversal of $\mathcal{F}_{3m}$. For $i, j \in \mathbb{Z}_3$, define $x_{ij}, y_{ij}, v_{ij}, w_{ij}$ to be the number of entries in T contained in the corresponding subrectangles as shown above. Further, let

$$u_{ij} = v_{ij} + w_{ij}, \quad \text{for } i = 0, 2 \text{ and } j \in \mathbb{Z}_3.$$

As in Section 3, a number of constraints are immediate from the definition of these parameters. We will make repeated implicit use of the bounds $0 \leqslant x_{11} \leqslant m$ and $0 \leqslant y_{ij}, u_{ij}, v_{ij}, w_{ij} \leqslant 1$. Moreover, the distribution of these parameters forces

$$\sum_{j\in\mathbb{Z}_3} (y_{ij} + u_{ij}) = 1, \quad \text{for } i \in \{0, 2\}, \tag{14}$$

$$y_{10} + y_{12} \leqslant 1, \qquad y_{10} + y_{12} + x_{11} \geqslant 1 \quad \text{and} \tag{15}$$

$$0 \leqslant x_{ij} \leqslant m - 1 \quad \text{if } (i, j) \neq (1, 1). \tag{16}$$

Also, the need for T to include one representative from each row, column and symbol of $\mathcal{F}_{3m}$ implies

$$x_{00} + x_{01} + x_{02} + y_{00} + y_{01} + y_{02} + u_{00} + u_{01} + u_{02} = m, \tag{17}$$

$$x_{10} + x_{11} + x_{12} + y_{10} + y_{12} = m, \tag{18}$$

$$x_{20} + x_{21} + x_{22} + y_{20} + y_{21} + y_{22} + u_{20} + u_{21} + u_{22} = m, \tag{19}$$

$$x_{00} + x_{10} + x_{20} + y_{00} + y_{10} + y_{20} + u_{00} + u_{20} = m, \tag{20}$$

$$x_{01} + x_{11} + x_{21} + y_{01} + y_{21} + u_{01} + u_{21} = m, \tag{21}$$

$$x_{02} + x_{12} + x_{22} + y_{02} + y_{12} + y_{22} + u_{02} + u_{22} = m, \tag{22}$$

$$x_{02} + x_{11} + x_{20} + y_{00} + y_{22} + u_{02} + u_{20} = m. \tag{23}$$

Adding (17), (19), then subtracting (21) and (23), gives

$$2x_{11} = x_{00} + x_{22} + y_{02} + y_{20} + u_{00} + u_{22}. \tag{24}$$

Moreover, Lemma 2.1 necessitates that

$$3 \mid u_{00} + 2y_{00} + 2u_{01} + y_{02} + 2y_{10} + y_{12} + 2y_{20} + u_{21} + 2u_{22} + y_{22}, \tag{25}$$

$$m \mid -x_{00} - x_{22} + y_{02} - 2y_{10} - 2y_{12} + y_{20} + u_{00} + u_{22}. \tag{26}$$

The above restrictions will help to show that entries of $\mathcal{F}_{3m}$ contained in the shaded regions in Fig. 2 are all transversal-free. To prove this, we need to generalize the definition of transversal to rectangular matrices with more columns than rows. A transversal of such a matrix is a selection of entries such that one entry is selected from each row and the chosen entries are in different columns and contain different symbols. We have the following lemma.

Lemma 4.3. *Suppose that m is odd. Let $M = [m_{ij}]$ be an $(m - 1) \times m$ matrix over $\mathbb{Z}_m$ where the rows are indexed by $\mathbb{Z}_m \setminus \{\ell\}$, the columns are indexed by $\mathbb{Z}_m$, and $m_{ij} = i + j - 1$. Suppose that T' is a transversal of M that hits every symbol except s and every column except c. Then $c \equiv s + 2 \bmod m$.*

Proof. All calculations will be in $\mathbb{Z}_m$. The sum of the symbols in T' is

$$\sum_{k\in\mathbb{Z}_m} k - s = \sum_{(i,j,k)\in T'} (i + j - 1) = \sum_{i\in\mathbb{Z}_m\setminus\{\ell\}} i + \sum_{j\in\mathbb{Z}_m} j - c - (m - 1)$$
$$= \sum_{i\in\mathbb{Z}_m} i + 1 + \sum_{j\in\mathbb{Z}_m} j - c + 1.$$

The result follows, since the sum of the elements of $\mathbb{Z}_m$ is 0. □

For $i \in \{0, 2\}$, let I_i be the subrectangle of size $(m - 1) \times m$ formed by rows $\{i0, i1, \ldots, ik\}$ and columns $\{i0, i1, \ldots, i\ell\}$. We will apply Lemma 4.3 to I_0 and I_2 several times in our next lemma (by projecting onto the second coordinate of $\mathbb{Z}_3 \oplus \mathbb{Z}_m$). Our aim is to demonstrate that $\mathcal{F}_{3m}$ has many transversal-free entries in rows 0ℓ and 2ℓ.

Lemma 4.4. $y_{00} + v_{01} = y_{01} + v_{02} = v_{20} + y_{21} = v_{21} + y_{22} = 0$.

Proof. By Lemma 4.2, we only need to prove that $y_{00} + v_{01} = y_{01} + v_{02} = 0$.

Assume that $y_{00}+v_{01}+y_{01}+v_{02} = 1$. By (14), it follows that $u_{00} = y_{02} = 0$. Now, (24) implies that $x_{00} + x_{22} + y_{20} + u_{22} = 2x_{11}$, which must be even. To satisfy (14), (16) and (26), the only possibilities are

(i) $y_{10} + y_{12} = 0, y_{20} + u_{22} = 0, x_{00} + x_{22} = 0$;
(ii) $y_{10} + y_{12} = 0, y_{20} + u_{22} = 1, x_{00} + x_{22} = 1$;
(iii) $y_{10} + y_{12} = 1, y_{20} + u_{22} = 0, x_{00} + x_{22} = 2(m - 1)$.

For (i), $x_{11} = 0$ by (24), which violates (15). Meanwhile, (ii) is incompatible with (14) and (25). So we may assume (iii).

We have $x_{ii} = m - 1$ for each $i \in \mathbb{Z}_3$ by (16) and (24), which means all $x_{ij} = 0$ when $i \neq j$ by (14) and (17)–(19). Now, in order to satisfy $y_{00} + v_{01} + y_{01} + v_{02} = 1$ there are four possibilities according to which variable is positive. The treatment of each case is similar.

Case 1: $y_{00} = 1$.

We deduce in turn that $u_{01} = y_{01} = 0$ by (14); $y_{12} = 1$ and $u_{21} + y_{22} = 0$ by (25) and $y_{21} = 1$ by (21). Since $x_{00} = m - 1$, then all the symbols $\{0\} \oplus \mathbb{Z}_m$ in T lie in I_0 and a subrectangle R formed by row 2ℓ and columns in $\{1\} \oplus Y_1$. Hence T includes a transversal T' of I_0 that misses the symbol that T hits in R. The symbols in R are $\{0\} \oplus (Y_1 - 1) = \{0\} \oplus Y_0$, so by Lemma 4.3, T' hits every column in I_0 except one column from $\{0\} \oplus (Y_0 + 2) = \{0\} \oplus V_0$. This means T' hits every column in $\{0\} \oplus Y_0$, which is a contradiction since $y_{00} = 1$.

Case 2: $v_{01} = 1$.

We deduce that $y_{00} = 0$ and $u_{01} = 1$ by (14); $y_{12} = 1, y_{10} = 0$ and $u_{21} + y_{22} = 0$ by (25); and $u_{20} = 1$ by (20). All the symbols $\{0\}\oplus\mathbb{Z}_m$ in T lie in I_0 and a subrectangle formed by row 0ℓ and columns in $\{1\} \oplus V_1$. Hence the transversal T' in I_0 is missing some symbol in $\{0\} \oplus (V_1 - 1) = \{0\} \oplus (Y_0 - 2)$. By Lemma 4.3, T' hits every column in I_0 except one column from $\{0\} \oplus Y_0$, which means T' must hit every column in $\{0\} \oplus U_0$. This is a contradiction since $u_{20} = 1$.

Case 3: $y_{01} = 1$.

We deduce that $y_{00} = u_{01} = u_{02} = 0$ by (14); $y_{12} = 0$ by (25) and $y_{22} = 1$ by (22). The symbols $\{1\} \oplus \mathbb{Z}_m$ in T lie in I_2 and a subrectangle formed by row 0ℓ and columns in $\{1\} \oplus Y_1$. Hence the transversal T' in I_2 is missing some symbol in $\{1\} \oplus (Y_1 - 1) = \{1\} \oplus Y_2$. By Lemma 4.3, T' hits every column in I_2 except one column from $\{2\} \oplus (Y_2 + 2) = \{2\} \oplus V_2$, which means T' must hit every column in $\{2\} \oplus Y_2$. This contradicts $y_{22} = 1$.

Case 4: $v_{02} = 1$.

We deduce that $y_{00} = u_{01} = y_{01} = 0$ by (14); $u_{21} + y_{22} = 1$ by (25) and so $u_{21} = 1$ by (21). The symbols $\{1\} \oplus \mathbb{Z}_m$ in T lie in I_2 and a subrectangle formed by row 2ℓ and columns in $\{1\} \oplus U_1$. Hence the transversal T' in I_2 is missing some symbol in $\{1\} \oplus (U_1 - 1)$. By Lemma 4.3, T' hits every column in I_2 except one column from $\{2\} \oplus (U_1 + 1)$. By Lemma 4.1(i), it follows that T' hits every column in $\{2\} \oplus V_2$ in I_2. This contradicts $v_{02} = 1$.

Since all options have led to a contradiction, the result is proved. □

4.3. Applications of $\mathcal{F}_{3m}$

Lemma 4.4 says that $\mathcal{F}_{3m}$ has $2(m-1)$ transversal-free entries in row 0ℓ (and similarly, in row 2ℓ). Hence, $\mathcal{F}_{3m}$ has at most $m+2$ disjoint transversals, which greatly improves the bound in Theorem 1.4 for odd orders which are divisible by 3.

Theorem 4.5. *If n is an odd multiple of 3, then $\mu(n) \leqslant n/3 + 2$.*

The following lemma shows further restrictions on the transversal entries in rows 0ℓ and 2ℓ.

Lemma 4.6. *In $\mathcal{F}_{3m}$, we have*

(i) $y_{02} + u_{00} = 1$ *if and only if* $y_{20} + u_{22} = 1$;
(ii) $w_{01} = 1$ *if and only if* $w_{20} = 1$;
(iii) $w_{21} = 1$ *if and only if* $w_{02} = 1$.

Proof. First, suppose that $y_{02} + u_{00} + y_{20} + u_{22} = 1$. By (24), $x_{00} + x_{22}$ is odd. So the only possibility satisfying (14), (16) and (26) is that $y_{10} + y_{12} = 0$, $x_{00} + x_{22} = 1$. In this case, the right hand side of (25) will become $u_{00} + y_{02} + 2y_{20} + 2u_{22} + 2w_{01} + w_{21}$, which will be $1 + w_{21}$ if $u_{00} + y_{02} = 1$, or $2 + 2w_{01}$ if $y_{20} + u_{22} = 1$. However, neither of these options can be divisible by 3 as required. Hence $y_{02} + u_{00} + y_{20} + u_{22} \neq 1$, which proves (i).

For (ii), if $w_{01} = 1$ and $w_{20} = 0$, then $w_{21} = w_{02} = 0$; if $w_{01} = 0$ and $w_{20} = 1$, then $w_{21} = 0$, and further $w_{02} = 0$ since $\mathcal{F}_{3m}[(0, \ell), (2, d)] = \mathcal{F}_{3m}[(2, \ell), (0, d)] = (2, d - 1)$ when $d \in W_0$. Hence, for both cases, we have that $y_{02} + u_{00} + y_{20} + u_{22} = 1$, which is a contradiction.

Result (iii) follows from (ii) by Lemma 4.2. □

We remark that parts (ii) and (iii) of Lemma 4.6 demonstrate that $\mathcal{F}_{3m}$ contains pairs of entries such that any transversal that includes one element of the pair necessarily includes the other as well. This property was called "crimped" in [5], where it was used to create Latin squares that have an orthogonal mate but are not in any triple of MOLS. However, $\mathcal{F}_{3m}$ clearly does not have an orthogonal mate, since it contains transversal-free entries.

Lemma 4.6 also allows us to construct a set S of permutations over $\mathbb{Z}_3 \oplus \mathbb{Z}_m$ with $|S| = 3m + 3$ and $\mathrm{cr}(S) \leqslant 3m - 2$. This gives a worse upper bound for $f(3m, 2)$ than Theorem 3.3, but the bound is still $n + O(1)$. We take as S the rows of $\mathcal{F}_{3m}$, plus a permutation

$$(1, 0)(1, 1) \cdots (1, \ell)(2, 0)(2, 1) \cdots (2, \ell)(0, 0)(0, 1) \cdots (0, \ell)$$

and any two other permutations g_1 and g_2 that satisfy $(0, d_0)^{g_1} = (2, d_0)$, $(1, d_1)^{g_1} = (0, d_1)$, $(1, d_1)^{g_2} = (1, d_1)$ and $(2, d_2)^{g_2} = (2, d_2)$, for $d_i \in W_i$ and $i \in \mathbb{Z}_3$.

5. Covering radius $n - 3$

An interesting special case of the covering radius problem is to insist that your set of permutations is in fact a permutation group. In this short section we find the covering radius of AGL(1, q), the affine general linear group of degree 1. We also find the value of $f(5, 3)$, which is the smallest nontrivial case for $s = 3$.

While the rest of the paper deals with the case $s = 2$, this section pertains to $s = 3$; that is $f(n, 3)$. In [4] it was shown that

- $\mathrm{cr}(\mathrm{AGL}(1, q)) = q - 2$ when q is even,
- $q - 4 \leqslant \mathrm{cr}(\mathrm{AGL}(1, q)) \leqslant q - 3$ when q is odd,
- $\mathrm{cr}(\mathrm{AGL}(1, q)) = q - 3$ when $q \not\equiv 1 \bmod 6$,

and it was conjectured that $\mathrm{cr}(\mathrm{AGL}(1, q)) = q - 3$ whenever q is odd. We now prove this conjecture. The proof was communicated to us by Simon Blackburn (Royal Holloway, University of London).

Theorem 5.1. $\mathrm{cr}(\mathrm{AGL}(1, q)) = q - 3$ *when q is odd.*

Proof. Let $\mathbb{F}_q$ be a field of odd order q. Given the previous results, it suffices to find a permutation that is at distance at least $q - 3$ from every permutation in AGL(1, q). Let $f : \mathbb{F}_q \to \mathbb{F}_q$ be defined by $f(0) = 0$ and $f(x) = 1/x$ for $x \neq 0$. Clearly f is a permutation.

Let g be a typical element of AGL(1, q), so $g(x) = ax + b$ for constants a and b with $a \neq 0$. When $x \neq 0$, we have $1/x = f(x) = g(x) = ax + b$ if and only if $ax^2 + bx - 1 = 0$. This quadratic has at most 2 solutions in $\mathbb{F}_q \setminus \{0\}$. So there are at most 3 solutions to $f(x) = g(x)$ in $\mathbb{F}_q$. So f has the required property; it is not within distance $q - 4$ of any member of AGL(1, q). □

A corollary is that $f(q, 3) \leqslant q(q-1)$ for all prime powers q. In particular, $f(5, 3) \leqslant 20$, as was noted in [4]. However, it is possible to do much better. The following set of permutations of $\mathbb{Z}_5$ shows that $f(5, 3) \leqslant 15$:

01234	43012	42301	30142	21403
14023	02134	10342	13240	23401
24310	32410	34021	40213	41320.

For large q, using AGL(1, q) is far from optimal since it has quadratically many members. It was shown in [4] that $f(n, s) = O(n \log n)$ for any fixed value of s. We end by showing that the above set of 15 permutations is optimal. In other words, $f(5, 3) = 15$.

For each $g \in \mathcal{S}_5$, let $B(g)$ be the ball of radius 2 centred at g. Let $B(S) = \cup_{g \in S} B(g)$ for any $S \subset \mathcal{S}_5$. The proof proceeds from the following three computational results.

(1) There does not exist a set $S \subset \mathcal{S}_5$ with $|S| = 14$, $\mathrm{cr}(S) = 2$, and in which 4 members have pairwise disjoint balls of radius 2.

(2) For any $g \in \mathcal{S}_5$ let $I(i)$ denote $\max |B(g) \setminus B(P)|$, where the maximum runs over all sets $P \subset \mathcal{S}_5 \setminus \{g\}$ of cardinality i such that $B(g) \cap B(p) \neq \emptyset$ for $p \in P$. Note that $I(i)$ is independent of g and it suffices to compute it when g is the identity. We found that $I(0) = 11$, $I(1) = 9$ and $I(2) = I(3) = 8$. Moreover, for all pairs (g, P) achieving $I(3) = 8$, $\{g\} \cup P$ is isomorphic to {01234, 02134, 01243, 02143}.

(3) There does not exist a set $S \subset \mathcal{S}_5$ with $|S| = 14$, $\mathrm{cr}(S) = 2$ and containing the four permutations 01234, 02134, 01243, 02143.

Results (1) and (3) were obtained by starting with a set of 4 permutations and extending it in all possible ways by backtracking. The other 10 permutations were chosen in a lexicographic order. The search was pruned whenever the number of permutations that remained uncovered was too great to be covered even if all subsequently chosen balls turned out to be disjoint.

We now deduce the nonexistence of a set $S \subset \mathcal{S}_5$ with $|S| = 14$ and $\mathrm{cr}(S) = 2$. Suppose such a set S exists. We seek disjoint subsets S_1, S_2, S_3 of S of cardinality 3 with the following property: for $1 \leqslant i \leqslant 3$ and for all $p \in S \setminus \cup_{j \leqslant i} S_j$, there exists some $p_j \in S_j$, for each $j \leqslant i$, such that $B(p) \cap B(p_j) \neq \emptyset$. By result (1), we may find each S_i in turn by including within it a subset of $S \setminus \cup_{j<i} S_j$ which is maximal with respect to the requirement that the corresponding balls of radius 2 are pairwise disjoint. Now,

$$\left|B(S_1 \cup S_2 \cup S_3)\right| \leqslant 3\big(I(0) + I(1) + I(2)\big) = 3(11 + 9 + 8) = 84,$$

by our choice of S_1, S_2, S_3. Also, we may assume that for each $p \in S \setminus (S_1 \cup S_2 \cup S_3)$ there exists $p_i \in S_i$, for $1 \leqslant i \leqslant 3$, such that $B(p) \cap B(p_i) \neq \emptyset$. Moreover, by results (2) and (3) above, we can assume that $\left|B(p) \setminus B(\{p_1, p_2, p_3\})\right| \leqslant 7$. Hence $|B(S)| \leqslant 84 + 5 \times 7 = 119 < 120$, which is a contradiction. We conclude that $f(5, 3) = 15$ as claimed.

6. Concluding remarks

We have improved the upper bounds on $f(n, 2)$, the size of the smallest set S of permutations in $\mathcal{S}_n$ such that every other permutation in $\mathcal{S}_n$ agrees with at least one element of S in at least two places. In particular, we showed that $f(n, 2) \leqslant n + O(\log n)$ for all n and $f(n, 2) \leqslant n + 2$ if $n > 3$ is divisible by 3. Further, in a Latin square L let $\lambda(L)$ be the maximum number of disjoint transversals and let $\mu(n)$ be the minimum value of λ among all the Latin squares of order n. We showed that $\mu(n) \leqslant \frac{1}{3}n + 2$ for all odd n divisible by 3, significantly improving on the previous best bound of $\mu(n) \leqslant \frac{1}{2}(n + 1)$.

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